

# NONLINEAR KLEIN-GORDON-MAXWELL SYSTEMS WITH NEUMANN BOUNDARY CONDITIONS ON A RIEMANNIAN MANIFOLD WITH BOUNDARY

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**ABSTRACT.** Let  $(M, g)$  be a smooth compact,  $n$  dimensional Riemannian manifold,  $n = 3, 4$  with smooth  $n - 1$  dimensional boundary  $\partial M$ . We search the positive solutions of the singularly perturbed Klein Gordon Maxwell Proca system with homogeneous Neumann boundary conditions or for the singularly perturbed Klein Gordon Maxwell system with mixed Dirichlet Neumann homogeneous boundary conditions. We prove that  $C^1$  stable critical points of the mean curvature of the boundary generates  $H^1(M)$  solutions when the perturbation parameter  $\varepsilon$  is sufficiently small.

## 1. INTRODUCTION

Let  $(M, g)$  be a smooth compact,  $n$  dimensional Riemannian manifold,  $n = 3, 4$  with boundary  $\partial M$  which is the union of a finite number of connected, smooth, boundaryless,  $n - 1$  submanifolds embedded in  $M$ . Here  $g$  denotes the Riemannian metric tensor. By Nash theorem we can consider  $(M, g)$  as a regular submanifold embedded in  $\mathbb{R}^N$ .

We search the positive solutions of the following Klein Gordon Maxwell Proca system with homogeneous Neumann boundary conditions

$$(1) \quad \begin{cases} -\varepsilon^2 \Delta_g u + au = |u|^{p-2}u + \omega^2(qv - 1)^2u & \text{in } M \\ -\Delta_g v + (1 + q^2 u^2)v = qu^2 & \text{in } M \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial M \end{cases}$$

or Klein Gordon Maxwell system with mixed Dirichlet Neumann homogeneous boundary conditions

$$(2) \quad \begin{cases} -\varepsilon^2 \Delta_g u + au = |u|^{p-2}u + \omega^2(qv - 1)^2u & \text{in } M \\ -\Delta_g v + q^2 u^2 v = qu^2 & \text{in } M \\ v = 0 & \text{on } \partial M \\ \frac{\partial u}{\partial \nu} = 0. & \text{on } \partial M \end{cases}$$

Here  $2 < p < 2^* = \frac{2n}{n-2}$ ,  $\nu$  is the external normal to  $\partial M$ ,  $a > 0$ ,  $q > 0$ ,  $\omega \in (-\sqrt{a}, \sqrt{a})$  and  $\varepsilon$  is a positive perturbation parameter.

We are interested in finding solutions  $u, v \in H_g^1(M)$  to problem (1) and (2). Also, we show that, for  $\varepsilon$  sufficiently small, the function  $u$  has a peak near a stable critical point of the mean curvature of the boundary.

2010 *Mathematics Subject Classification.* 35J57, 35J60, 58E05, 81V10.

*Key words and phrases.* Riemannian manifold with boundary, Klein Gordon Maxwell systems, Neumann boundary condition, Mean curvature, Liapounov Schmidt.

The first author was supported by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of Istituto Nazionale di Alta Matematica (INdAM).

**Definition 1.** Let  $f \in C^1(N, \mathbb{R})$ , where  $(N, g)$  is a Riemannian manifold. We say that  $K \subset N$  is a  $C^1$ -stable critical set of  $f$  if  $K \subset \{x \in N : \nabla_g f(x) = 0\}$  and for any  $\mu > 0$  there exists  $\delta > 0$  such that, if  $h \in C^1(N, \mathbb{R})$  with

$$\max_{d_g(x, K) \leq \mu} |f(x) - h(x)| + |\nabla_g f(x) - \nabla_g h(x)| \leq \delta,$$

then  $h$  has a critical point  $x_0$  with  $d_g(x_0, K) \leq \mu$ . Here  $d_g$  denotes the geodesic distance associated to the Riemannian metric  $g$ .

Now we state the main theorem.

**Theorem 2.** Assume  $K \subset \partial M$  is a  $C^1$ -stable critical set of the mean curvature of the boundary. Then there exists  $\varepsilon_0 > 0$  such that, for any  $\varepsilon \in (0, \varepsilon_0)$ , Problem (1) has a solution  $(u_\varepsilon, v_\varepsilon) \in H_g^1(M) \times H_g^1(M)$ . Analogously, problem (2) has a solution  $(u_\varepsilon, v_\varepsilon) \in H_g^1(M) \times H_{0,g}^1(M)$ . Moreover, the function  $u_\varepsilon$  has a peak in some  $\xi_\varepsilon \in \partial M$  which converges to a point  $\xi_0 \in K$  as  $\varepsilon$  goes to zero.

From the seminal paper of [2] many authors studied KGM systems on a flat domain. We cite [1, 4, 6, 7, 8, 9, 10, 21].

For KGM and KGMP system on Riemannian manifolds, as far as we know the first paper in which deals with this problem is by Druet and Hebey [11]. In this work the authors study the case  $\varepsilon = 1$  and prove the existence of a solution for KGMP systems on a closed manifold, by the mountain pass theorem. Thereafter several works are devoted to the study of KGMP system on Riemannian closed manifold. We limit ourself to cite [18, 19, 5, 16, 17].

Klein Gordon Maxwell system provides a model for a particle  $u$  interacting with its own electrostatic field  $v$ . Thus, is somewhat more natural to prescribe Neumann condition on the second equation as d'Avenia Pisani and Siciliano nicely explained in the introduction of [8].

So, recently we moved to study KGMP systems in a Riemannian manifold  $M$  with boundary  $\partial M$  with Neumann boundary condition on the second equation. In [14] the authors proved that the topological properties of the boundary  $\partial M$ , namely the Lusternik Schnirelmann category of the boundary, affects the number of the low energy solution for the systems. Also, we notice that the natural dimension for KGM and KGMP systems is  $n = 3$ , since this systems arises from a physical model. However, the case  $n = 4$  is interesting from a mathematical point of view, since the second equation of systems (1) and (2) becomes energy critical by the presence of the  $u^2 v$  term. For further comments on this subject, we refer to [18].

We can compare [14] and Theorem 2. In [15] we proved that the set of metrics for which the mean curvature has only nondegenerate critical points is an open dense set among all the  $C^k$  metrics on  $M$ ,  $k \geq 3$ . Thus, generically with respect to the metric, the mean curvature has  $P_1(\partial M)$  nondegenerate (hence stable) critical points, where  $P_1(\partial M)$  is the Poincaré polynomial of  $\partial M$ , namely  $P_t(\partial M)$ , evaluated in  $t = 1$ . Hence, generically with respect to metric, Problem (1) has  $P_1(\partial M)$  solution and holds  $P_1(\partial M) \geq \text{cat} \partial M$ . Also, in many cases the strict inequality  $P_1(\partial M) > \text{cat} \partial M$  holds.

The paper is organized as follows. In Section 2 we summarize some result that are necessary to frame the problem. Namely, we recall some well known notion of Riemannian geometry, we introduce the variational setting and we study some properties of the second equation of the systems. In Section 3 we perform the

finite dimensional reduction and we sketch the prove of Theorem 2. A collection of technical results is contained in Appendix A.

## 2. PRELIMINARY RESULTS

We recall some well known fact about Riemannian manifold with boundary.

First of all we define the Fermi coordinate chart.

**Definition 3.** If  $q$  belongs to the boundary  $\partial M$ , let  $\bar{y} = (z_1, \dots, z_{n-1})$  be Riemannian normal coordinates on the  $n-1$  manifold  $\partial M$  at the point  $q$ . For a point  $\xi \in M$  close to  $q$ , there exists a unique  $\bar{\xi} \in \partial M$  such that  $d_g(\xi, \partial M) = d_g(\xi, \bar{\xi})$ . We set  $\bar{z}(\xi) \in \mathbb{R}^{n-1}$  the normal coordinates for  $\bar{\xi}$  and  $z_n(\xi) = d_g(\xi, \partial M)$ . Then we define a chart  $\Psi_q^\partial : \mathbb{R}_+^n \rightarrow M$  such that  $(\bar{z}(\xi), z_n(\xi)) = (\Psi_q^\partial)^{-1}(\xi)$ . These coordinates are called *Fermi coordinates* at  $q \in \partial M$ . The Riemannian metric  $g_q(\bar{z}, z_n)$  read through the Fermi coordinates satisfies  $g_q(0) = \text{Id}$ .

We note by  $d_g^\partial$  and  $\exp^\partial$  respectively the geodesic distance and the exponential map on by  $\partial M$ . By compactness of  $\partial M$ , there is an  $R^\partial$  and a finite number of points  $q_i \in \partial M$ ,  $i = 1, \dots, k$  such that

$$I_{q_i}(R^\partial, R_M) := \{x \in M, d_g(x, \partial M) = d_g(x, \bar{\xi}) < R_M, d_g^\partial(q_i, \bar{\xi}) < R^\partial\}$$

form a covering of  $(\partial M)_\rho$  and on every  $I_{q_i}$  the Fermi coordinates are well defined. In the following we choose,  $R = \min\{R^\partial, R_M\}$ , such that we have a finite covering

$$M \subset \{\cup_{i=1}^k B(q_i, R)\} \cup \{\cup_{i=k+1}^l I_{\xi_i}(R, R)\}$$

where  $k, l \in \mathbb{N}$ ,  $q_i \in M \setminus \partial M$  and  $\xi_i \in \partial M$ .

Given the Fermi coordinates in a neighborhood of  $p$ , and we denoted by the matrix  $(h_{ij})_{i,j=1,\dots,n-1}$  the second fundamental form, we have the well known formulas (see [3, 12])

$$(3) \quad g^{ij}(y) = \delta_{ij} + 2h_{ij}(0)y_n + O(|y|^2) \text{ for } i, j = 1, \dots, n-1$$

$$(4) \quad g^{in}(y) = \delta_{in}$$

$$(5) \quad \sqrt{g}(y) = 1 - (n-1)H(0)y_n + O(|y|^2)$$

where  $(y_1, \dots, y_n)$  are the Fermi coordinates and the mean curvature  $H$  is

$$(6) \quad H = \frac{1}{n-1} \sum_i^{n-1} h_{ii}$$

To solve our system, using an idea of Benci and Fortunato [2], we reduce the system to a single equation. We introduce the map  $\psi$  defined by the equation

$$(7) \quad \begin{cases} -\Delta_g \psi + (1 + q^2 u^2) \psi = qu^2 & \text{in } M \\ \frac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial M \end{cases}$$

in case of Neumann boundary condition or by

$$(8) \quad \begin{cases} -\Delta_g \psi + qu^2 \psi = qu^2 & \text{in } M \\ \psi = 0 & \text{on } \partial M \end{cases}$$

in case of Dirichlet boundary condition.

In what follows we call  $H = H_g^1$  for the Neumann problem and  $H = H_{0,g}^1$  for the Dirichlet problem. Thus with abuse of language we will say that  $\psi : H \rightarrow H$

in both (7) and (8). Moreover, from standard variational arguments, it is easy to see that  $\psi$  is well-defined in  $H$  and it holds

$$(9) \quad 0 \leq \psi(u) \leq 1/q$$

for all  $u \in H$ . We collect now some well known result on the map  $\psi$ . For a more extensive presentation of these properties we refer to [11]

**Lemma 4.** *The map  $\psi : H \rightarrow H$  is  $C^2$  and its differential  $\psi'(u)[h] = V_u[h]$  at  $u$  is the map defined by*

$$(10) \quad -\Delta_g V_u[h] + (1 + q^2 u^2) V_u[h] = 2qu(1 - q\psi(u))h \text{ for all } h \in H.$$

*in case of Neumann boundary condition or*

$$(11) \quad -\Delta_g V_u[h] + q^2 u^2 V_u[h] = 2qu(1 - q\psi(u))h \text{ for all } h \in H.$$

*in case of Dirichlet boundary condition.*

*Also, we have*

$$0 \leq \psi'(u)[u] \leq \frac{2}{q}.$$

*Finally, the second derivative  $(h, k) \rightarrow \psi''(u)[h, k] = T_u(h, k)$  is the map defined by the equation*

$$-\Delta_g T_u(h, k) + (1 + q^2 u^2) T_u(h, k) = -2q^2 u(k V_u(h) + h V_u(k)) + 2q(1 - q\psi(u))hk$$

*in case of Neumann boundary condition or*

$$-\Delta_g T_u(h, k) + q^2 u^2 T_u(h, k) = -2q^2 u(k V_u(h) + h V_u(k)) + 2q(1 - q\psi(u))hk$$

*in case of Dirichlet boundary condition.*

**Lemma 5.** *The map  $\Theta : H \rightarrow \mathbb{R}$  given by*

$$\Theta(u) = \frac{1}{2} \int_M (1 - q\psi(u)) u^2 d\mu_g$$

*is  $C^2$  and*

$$\Theta'(u)[h] = \int_M (1 - q\psi(u))^2 u h d\mu_g$$

*for any  $u, h \in H$*

For the proofs of these results we refer to [11], in which the case of KGMP is treated. For KGM systems, the proof is identical.

Now, we introduce the functionals  $I_\varepsilon, J_\varepsilon, G_\varepsilon : H \rightarrow \mathbb{R}$

$$(12) \quad I_\varepsilon(u) = J_\varepsilon(u) + \frac{\omega^2}{2} G_\varepsilon(u),$$

where

$$(13) \quad J_\varepsilon(u) := \frac{1}{2\varepsilon^n} \int_M [\varepsilon^2 |\nabla_g u|^2 + (a - \omega^2) u^2] d\mu_g - \frac{1}{p\varepsilon^n} \int_M (u^+)^p d\mu_g$$

and

$$(14) \quad G_\varepsilon(u) := \frac{1}{\varepsilon^n} q \int_M \psi(u) u^2 d\mu_g.$$

By Lemma 5 we deduce that

$$(15) \quad \frac{1}{2} G'_\varepsilon(u)[\varphi] = \frac{1}{\varepsilon^n} \int_M [2q\psi(u) - q^2 \psi^2(u)] u \varphi d\mu_g.$$

If  $u \in H$  is a critical point of  $I_\varepsilon$  then the pair  $(u, \psi(u))$  is the desired solution of Problem (1) or (2).

Finally, we introduce a model function for the solution  $u$ . It is well known that, in  $\mathbb{R}^n$ , there is a unique positive radially symmetric function  $V(z) \in H^1(\mathbb{R}^n)$  satisfying

$$(16) \quad -\Delta V + (a - \omega^2)V = V^{p-1} \text{ on } \mathbb{R}^n.$$

Moreover, the function  $V$  exponentially decays at infinity as well as its derivative, that is, for some  $c > 0$

$$\lim_{|z| \rightarrow \infty} V(|z|)|z|^{\frac{n-1}{2}}e^{|z|} = c \quad \lim_{|z| \rightarrow \infty} V'(|z|)|z|^{\frac{n-1}{2}}e^{|z|} = -c.$$

We can define on the half space  $\mathbb{R}_+^n = \{(z_1, \dots, z_n) \in \mathbb{R}^n, z_n \geq 0\}$  the function

$$U(x) = V|_{x_n \geq 0}.$$

The function  $U$  satisfies the following Neumann problem in  $\mathbb{R}_+^n$

$$(17) \quad \begin{cases} -\Delta U + (a - \omega^2)U = U^{p-1} & \text{in } \mathbb{R}_+^n \\ \frac{\partial U}{\partial z_n} = 0 & \text{on } \{z_n = 0\}. \end{cases}$$

and it is easy to see that the space solution of the linearized problem

$$(18) \quad \begin{cases} -\Delta \varphi + (a - \omega^2)\varphi = (p-1)U^{p-2}\varphi & \text{in } \mathbb{R}_+^n \\ \frac{\partial \varphi}{\partial z_n} = 0 & \text{on } \{z_n = 0\}. \end{cases}$$

is generated by the linear combination of

$$\varphi^i = \frac{\partial U}{\partial z_i}(z) \text{ for } i = 1, \dots, n-1.$$

We endow  $H_g^1(M)$  with the scalar product  $\langle u, v \rangle_\varepsilon := \frac{1}{\varepsilon^n} \int_M \varepsilon^2 \nabla_g u \nabla_g v + (a - \omega^2)uv d\mu_g$

and the norm  $\|u\|_\varepsilon = \langle u, u \rangle_\varepsilon^{1/2}$ . We call  $H_\varepsilon$  the space  $H_g^1$  equipped with the norm  $\|\cdot\|_\varepsilon$ . We also define  $L_\varepsilon^p$  as the space  $L_g^p(M)$  endowed with the norm

$$\|u\|_{\varepsilon, p} = \frac{1}{\varepsilon^n} \left( \int_M u^p d\mu_g \right)^{1/p}.$$

For any  $p \in [2, 2^*)$ , the embedding  $i_\varepsilon : H_\varepsilon \hookrightarrow L_{\varepsilon, p}$  is a compact, continuous map, and it holds  $\|u\|_{\varepsilon, p} \leq c\|u\|_\varepsilon$  for some constant  $c$  not depending on  $\varepsilon$ . We define the adjoint operator  $i_\varepsilon^* : L_{\varepsilon, p'} \hookrightarrow H_\varepsilon$  as

$$u = i_\varepsilon^*(v) \Leftrightarrow \langle u, \varphi \rangle_\varepsilon = \frac{1}{\varepsilon^n} \int_M v \varphi d\mu_g.$$

Now on set

$$f(u) = |u^+|^{p-1}$$

and

$$g(u) := (q^2 \psi^2(u) - 2q\psi(u))u.$$

we can rewrite problem (1) in an equivalent formulation

$$u = i_\varepsilon^* [f(u) + \omega^2 g(u)], \quad u \in H_\varepsilon.$$

*Remark 6.* We have that  $\|i_\varepsilon^*(v)\|_\varepsilon \leq c\|v\|_{p', \varepsilon}$  with  $c$  independent by  $\varepsilon$ .

*Remark 7.* We recall the following two estimates, that can be obtained by trivial computations

$$(19) \quad \|u\|_{H_g^1} \leq c\varepsilon^{\frac{1}{2}} \|u\|_\varepsilon \text{ for } n = 3$$

$$(20) \quad \|u\|_{H_g^1} \leq c\varepsilon \|u\|_\varepsilon \text{ for } n = 4$$

We often will use the estimate (19) also when  $n = 4$ , which is still true even if weaker, to simplify the exposition.

Finally, we define an important class of functions on the manifold, modeled on the function  $U$ . For all  $\xi \in \partial M$  we define

$$W_{\varepsilon,\xi} = \begin{cases} U_\varepsilon \left( \left( \Psi_\xi^\partial \right)^{-1} (x) \right) \chi_R \left( \left( \Psi_\xi^\partial \right)^{-1} (x) \right) & x \in I_\xi(R) := I_\xi(R, R); \\ 0 & \text{elsewhere.} \end{cases}$$

We recall a fundamental limit property for the function  $W_{\varepsilon,\xi}$ .

*Remark 8.* Since  $U$  decays exponentially, it holds, uniformly with respect to  $q \in \partial M$ ,

$$(21) \quad \lim_{\varepsilon \rightarrow 0} |W_{\varepsilon,\xi}|_{t,\varepsilon}^t = \int_{\mathbb{R}_+^n} U^t(z) dz$$

for all  $1 \leq t \leq 2^*$ , and

$$(22) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^2 |\nabla_g W_{\varepsilon,\xi}|_{2,\varepsilon}^2 = \int_{\mathbb{R}_+^n} |\nabla U|^2(z) dz$$

We also have the following estimate for the function  $\psi$  and for its differential  $\psi'$ .

**Lemma 9.** *It holds, for any  $\varphi \in H$  and for any  $\xi \in \partial M$*

$$(23) \quad \|\psi(W_{\varepsilon,\xi} + \varphi)\|_H \leq c_1 \left( \varepsilon^{\frac{n+2}{2}} + \|\varphi\|_H^2 \right)$$

$$(24) \quad \|\psi(W_{\varepsilon,\xi} + \varphi)\|_H \leq c_2 \varepsilon^{\frac{n+2}{2}} (1 + \|\varphi\|_\varepsilon^2)$$

for some positive constants  $c_1, c_2$ , when  $\varepsilon$  is sufficiently small.

*Proof.* We prove the claim for the Neumann boundary condition. For the Dirichlet boundary condition the proof is completely analogous taking in account the gradient norm on  $H$ .

To simplify the notations we set  $v = \psi(W_{\varepsilon,\xi} + \varphi)$ . By definition of  $\psi$  we have

$$\begin{aligned} \|v\|_H^2 &\leq \int_M |\nabla_g v|^2 + v^2 + q^2 (W_{\varepsilon,\xi} + \varphi)^2 v^2 = q \int (W_{\varepsilon,\xi} + \varphi)^2 v \\ &\leq \left( \int_M v^{2^*} \right)^{\frac{1}{2^*}} \left( \int_M (W_{\varepsilon,\xi} + \varphi)^{\frac{4n}{n+2}} \right)^{\frac{n+2}{2n}} \leq c \|v\|_{H_g^1} |W_{\varepsilon,\xi} + \varphi|_{\frac{4n}{n+2},g}^2 \\ &\leq c \|v\|_{H_g^1} \left( |W_{\varepsilon,\xi}|_{\frac{4n}{n+2},g}^2 + |\varphi|_{\frac{4n}{n+2},g}^2 \right) \end{aligned}$$

Thus  $\|v\|_H \leq c \left( |W_{\varepsilon,\xi}|_{\frac{4n}{n+2},g} + |\varphi|_{\frac{4n}{n+2},g} \right)$ . Taking in account (21) of Remark 8 we have that, for  $\varepsilon$  small  $|W_{\varepsilon,\xi}|_{\frac{4n}{n+2},g}^2 \leq C \varepsilon^{\frac{2n}{n+2}} |U|_{\frac{4n}{n+2},g}^2$ . Thus we have

$$(25) \quad \|v\|_{H_g^1} \leq c_1 \left( \varepsilon^{\frac{2n}{n+2}} + |\varphi|_{\frac{4n}{n+2},g}^2 \right) \leq c_1 \left( \varepsilon^{\frac{2n}{n+2}} + \|\varphi\|_{H_g^1}^2 \right)$$

and

$$(26) \quad \|v\|_{H_g^1} \leq c_2 \varepsilon^{\frac{2n}{n+2}} \left(1 + |\varphi|_{\frac{2n}{n+2}, \varepsilon}^2\right) \leq c_2 \varepsilon^{\frac{2n}{n+2}} (1 + \|\varphi\|_{\varepsilon}^2).$$

that prove (23) and (24). For any  $\xi \in M$  and  $h, k \in H_g^1$  it holds  $\square$

**Lemma 10.** *It holds, for any  $h, k \in H$  and for any  $\xi \in \partial M$*

$$\|\psi'(W_{\varepsilon, \xi} + k)[h]\|_H \leq c \|h\|_H \{\varepsilon^2 + \|k\|_H\}$$

for some positive constant  $c$  when  $\varepsilon$  is sufficiently small.

*Proof.* Again, we prove the claim for the Neumann boundary condition being the other case completely analogous. By (10) and since  $0 < \psi < 1/q$ ,

$$\begin{aligned} \|\psi'(W_{\varepsilon, \xi} + k)[h]\|_{H_g^1}^2 &= 2q \int_M (W_{\varepsilon, \xi} + k)(1 - q\psi(W_{\varepsilon, \xi} + k))h\psi'(W_{\varepsilon, \xi} + k)[h] \\ &\quad - q^2 \int_M (W_{\varepsilon, \xi} + k)^2 (\psi'(W_{\varepsilon, \xi} + k)[h])^2 \\ &\leq \int_M W_{\varepsilon, \xi} |h| |\psi'(W_{\varepsilon, \xi} + k)[h]| + \int_M |k| |h| |\psi'(W_{\varepsilon, \xi} + k)[h]| \\ &:= I_1 + I_2 \end{aligned}$$

We estimate the two terms  $I_1$  and  $I_2$  separately. We have

$$I_1 \leq |\psi'(W_{\varepsilon, \xi} + k)[h]|_{2^*, g} |h|_{2^*, g} |W_{\varepsilon, \xi}|_{\frac{2}{n}, g} \leq \varepsilon^2 \|\psi'\|_{H_g^1} \|h\|_{H_g^1} |W_{\varepsilon, \xi}|_{\frac{n}{2}, \varepsilon}$$

$$I_2 \leq \|k\|_{L_g^3} \|h\|_{L_g^3} \|\psi'(W_{\varepsilon, \xi} + k)[h]\|_{L_g^3} \leq \|k\|_{H_g^1} \|h\|_{H_g^1} \|\psi'\|_{H_g^1}$$

and, in light of Remark 8, we obtain the claim.  $\square$

**2.1. The Lyapunov Schmidt reduction.** We want to split the space  $H_\varepsilon$  in a finite dimensional space generated by the solution of (18) and its orthogonal complement. Fixed  $\xi \in \partial M$  and  $R > 0$ , we consider on the manifold the functions

$$(27) \quad Z_{\varepsilon, \xi}^i = \begin{cases} \varphi_\varepsilon^i \left( \left( \psi_\xi^\partial \right)^{-1} (x) \right) \chi_R \left( \left( \psi_\xi^\partial \right)^{-1} (x) \right) & x \in I_\xi(R) := I_\xi(R, R); \\ 0 & \text{elsewhere.} \end{cases}$$

where  $\varphi_\varepsilon^i(z) = \varphi^i \left( \frac{z}{\varepsilon} \right)$  and  $\chi_R : B^{n-1}(0, R) \times [0, R) \rightarrow \mathbb{R}^+$  is a smooth cut off function such that  $\chi_R \equiv 1$  on  $B^{n-1}(0, R/2) \times [0, R/2)$  and  $|\nabla \chi| \leq 2$ .

In the following, for sake of simplicity, we denote

$$(28) \quad D^+(R) = B^{n-1}(0, R) \times [0, R) \subset \mathbb{R}_+^n$$

Let

$$K_{\varepsilon, \xi} := \text{Span} \left\{ Z_{\varepsilon, \xi}^1, \dots, Z_{\varepsilon, \xi}^{n-1} \right\}.$$

We can split  $H_\varepsilon$  in the sum of the  $(n-1)$ -dimensional space and its orthogonal complement with respect of  $\langle \cdot, \cdot \rangle_\varepsilon$ , i.e.

$$K_{\varepsilon, \xi}^\perp := \left\{ u \in H_\varepsilon, \langle u, Z_{\varepsilon, \xi}^i \rangle_\varepsilon = 0 \right\}.$$

We solve problem (1) by a Lyapunov Schmidt reduction: we look for a function of the form  $W_{\varepsilon, \xi} + \phi$  with  $\phi \in K_{\varepsilon, \xi}^\perp$  such that

$$(29) \quad \Pi_{\varepsilon, \xi}^\perp \left\{ W_{\varepsilon, \xi} + \phi - i_\varepsilon^* [f(W_{\varepsilon, \xi} + \phi) + \omega^2 g(W_{\varepsilon, \xi} + \phi)] \right\} = 0$$

$$(30) \quad \Pi_{\varepsilon, \xi} \left\{ W_{\varepsilon, \xi} + \phi - i_\varepsilon^* [f(W_{\varepsilon, \xi} + \phi) + \omega^2 g(W_{\varepsilon, \xi} + \phi)] \right\} = 0$$

where  $\Pi_{\varepsilon,\xi} : H_\varepsilon \rightarrow K_{\varepsilon,\xi}$  and  $\Pi_{\varepsilon,\xi}^\perp : H_\varepsilon \rightarrow K_{\varepsilon,\xi}^\perp$  are, respectively, the projection on  $K_{\varepsilon,\xi}$  and  $K_{\varepsilon,\xi}^\perp$ . We see that  $W_{\varepsilon,\xi} + \phi$  is a solution of (1) if and only if  $W_{\varepsilon,\xi} + \phi$  solves (29-30).

### 3. REDUCTION TO FINITE DIMENSIONAL SPACE

In this section we find a solution for equation (29). In particular, we prove that for all  $\varepsilon > 0$  and for all  $\xi \in \partial M$  there exists  $\phi_{\varepsilon,\xi} \in K_{\varepsilon,\xi}^\perp$  solving (29). The main part of the reduction is performed in [13] and in [20]. Here we explicitly estimate only the term appearing in this specific contest.

We can rewrite equation (29) as

$$L_{\varepsilon,\xi}(\phi) = N_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi} + S_{\varepsilon,\xi}(\phi)$$

where  $L_{\varepsilon,\xi}$  is the linear operator

$$\begin{aligned} L_{\varepsilon,\xi} & : K_{\varepsilon,\xi}^\perp \rightarrow K_{\varepsilon,\xi}^\perp \\ L_{\varepsilon,\xi}(\phi) & := \Pi_{\varepsilon,\xi}^\perp \{ \phi - i_\varepsilon^* [f'(W_{\varepsilon,\xi})\phi] \}, \end{aligned}$$

$N_{\varepsilon,\xi}(\phi)$  is the nonlinear term

$$N_{\varepsilon,\xi} := \Pi_{\varepsilon,\xi}^\perp \{ i_\varepsilon^* [f(W_{\varepsilon,\xi} + \phi) - f(W_{\varepsilon,\xi}) - f'(W_{\varepsilon,\xi})\phi] \}$$

$R_{\varepsilon,\xi}$  is a remainder term

$$R_{\varepsilon,\xi} := \Pi_{\varepsilon,\xi}^\perp \{ i_\varepsilon^* [f(W_{\varepsilon,\xi})] - W_{\varepsilon,\xi} \}$$

and  $S_{\varepsilon,\xi}$  is the coupling term

$$S_{\varepsilon,\xi} = \Pi_{\varepsilon,\xi}^\perp \{ i_\varepsilon^* [\omega^2 g(W_{\varepsilon,\xi} + \phi)] \}.$$

**Proposition 11.** *premise There exists  $\varepsilon_0 > 0$  and  $C > 0$  such that for any  $\xi \in \partial M$  and for all  $\varepsilon \in (0, \varepsilon_0)$  there exists a unique  $\phi_{\varepsilon,\xi} = \phi(\varepsilon, \xi) \in K_{\varepsilon,\xi}^\perp$  which solves (29). Moreover*

$$\|\phi_{\varepsilon,\xi}\|_\varepsilon < C\varepsilon^2.$$

Finally,  $\xi \mapsto \phi_{\varepsilon,\xi}$  is a  $C^1$  map.

To prove this result, we premise some technical lemma.

*Remark 12.* We summarize here the results on  $L_{\varepsilon,\xi}$ ,  $N_{\varepsilon,\xi}$  and  $R_{\varepsilon,\xi}$  contained in [13].

There exist  $\varepsilon_0$  and  $c > 0$  such that, for any  $\xi \in \partial M$  and  $\varepsilon \in (0, \varepsilon_0)$

$$\|L_{\varepsilon,\xi}\|_\varepsilon \geq c\|\phi\|_\varepsilon \text{ for any } \phi \in K_{\varepsilon,\xi}^\perp.$$

Also it holds

$$\|R_{\varepsilon,\xi}\|_\varepsilon \leq c\varepsilon^{1+\frac{n}{p^*}}$$

and

$$\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon \leq c(\|\phi\|_\varepsilon^2 + \|\phi\|_\varepsilon^{p-1})$$

We further remark that  $\frac{n}{p^*} > 1$  since  $2 \leq p < 2^*$

We have now to estimate the coupling term  $S_{\varepsilon,\xi}$ .

**Lemma 13.** *If  $\|\phi\|_\varepsilon, \|\phi_1\|_\varepsilon, \|\phi_2\|_\varepsilon = O(\varepsilon^2)$  it holds*

$$(31) \quad \|S_{\varepsilon,\xi}(\phi)\|_\varepsilon \leq c\varepsilon^2$$

$$(32) \quad \|S_{\varepsilon,\xi}(\phi_1) - S_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \leq l_\varepsilon \|\phi_1 - \phi_2\|_\varepsilon$$

where  $l_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .



*Proof.* We have, by the properties of the map  $i_\varepsilon^*$ , that

$$\begin{aligned}
\|S_{\varepsilon,\xi}(\phi)\|_\varepsilon &\leq c |\psi^2(W_{\varepsilon,\xi} + \phi)(W_{\varepsilon,\xi} + \phi)|_{\varepsilon,p'} + c |\psi(W_{\varepsilon,\xi} + \phi)(W_{\varepsilon,\xi} + \phi)|_{\varepsilon,p'} \\
&\leq c |\psi(W_{\varepsilon,\xi} + \phi)(W_{\varepsilon,\xi} + \phi)|_{\varepsilon,p'} \\
&\leq \frac{c}{\varepsilon^{\frac{n}{p'}}} \left( \int \psi(W_{\varepsilon,\xi} + \phi)^{2^*} \right)^{\frac{1}{2^*}} \left( \int |W_{\varepsilon,\xi} + \phi|^{p'(\frac{2^*}{p'})'} \right)^{\frac{1}{p'(\frac{2^*}{p'})'}} \\
&\leq c \varepsilon^{-\frac{n}{p'} + \frac{n}{p'(\frac{2^*}{p'})'}} \|\psi(W_{\varepsilon,\xi} + \phi)\|_H |W_{\varepsilon,\xi} + \phi|_{\varepsilon,p'(\frac{2^*}{p'})'} \\
&\leq c \varepsilon^{-\frac{n}{2^*}} \|\psi(W_{\varepsilon,\xi} + \phi)\|_H \leq c \varepsilon^{-\frac{n}{2^*}} \varepsilon^{\frac{n+2}{2}} = c \varepsilon^2
\end{aligned}$$

by (24) and taking in account that  $\|\phi\|_\varepsilon = o(1)$  by Remark 7, and the first step is proved.

For the second claim, we have, since  $0 \leq \psi \leq 1/q$

$$\begin{aligned}
\|S_{\varepsilon,\xi}(\phi_1) - S_{\varepsilon,\xi}(\phi_2)\|_\varepsilon &\leq c |\psi^2(W_{\varepsilon,\xi} + \phi_1)(W_{\varepsilon,\xi} + \phi_1) - \psi^2(W_{\varepsilon,\xi} + \phi_2)(W_{\varepsilon,\xi} + \phi_2)|_{\varepsilon,p'} \\
&\quad + c |\psi(W_{\varepsilon,\xi} + \phi_1)(W_{\varepsilon,\xi} + \phi_1) - \psi(W_{\varepsilon,\xi} + \phi_2)(W_{\varepsilon,\xi} + \phi_2)|_{\varepsilon,p'} \\
&\leq c |\psi(W_{\varepsilon,\xi} + \phi_1)(W_{\varepsilon,\xi} + \phi_1) - \psi(W_{\varepsilon,\xi} + \phi_2)(W_{\varepsilon,\xi} + \phi_2)|_{\varepsilon,p'} \\
&\leq c |[\psi(W_{\varepsilon,\xi} + \phi_1) - \psi(W_{\varepsilon,\xi} + \phi_2)](W_{\varepsilon,\xi} + \phi_1)|_{\varepsilon,p'} \\
&\quad + |\psi(W_{\varepsilon,\xi} + \phi_2)[\phi_1 - \phi_2]|_{\varepsilon,p'} \\
&\leq c |[\psi'(W_{\varepsilon,\xi} + (1-\theta)\phi_1 + \theta\phi_2)[\phi_1 - \phi_2]](W_{\varepsilon,\xi} + \phi_1)|_{\varepsilon,p'} \\
&\quad + |\psi(W_{\varepsilon,\xi} + \phi_2)[\phi_1 - \phi_2]|_{\varepsilon,p'} := D_1 + D_2
\end{aligned}$$

for some  $\theta \in (0, 1)$ . Arguing as in the first part of the proof we get, in light of (24), that

$$D_2 \leq c \varepsilon^{-\frac{n}{p^*}} \|\psi(W_{\varepsilon,\xi} + \phi)\|_H |\phi_1 - \phi_2|_{\varepsilon,p'(\frac{2^*}{p'})'} \leq c \varepsilon^{-\frac{n}{2^*}} \varepsilon^{\frac{n+2}{2}} \|\phi_1 - \phi_2\|_\varepsilon$$

and, using Lemma 10, that

$$\begin{aligned}
D_1 &\leq c \varepsilon^{-\frac{n}{p^*}} \|[\psi'(W_{\varepsilon,\xi} + (1-\theta)\phi_1 + \theta\phi_2)[\phi_1 - \phi_2]]\|_H |W_{\varepsilon,\xi} + \phi_1|_{\varepsilon,p'(\frac{2^*}{p'})'} \\
&\leq c \varepsilon^{-\frac{n}{p^*}} \{ \varepsilon^2 + (1-\theta)\|\phi_1\|_H + \theta\|\phi_2\|_H \} \|\phi_1 - \phi_2\|_H.
\end{aligned}$$

If  $n = 3$ , by (19) and since  $\|\phi_1\|_\varepsilon, \|\phi_2\|_\varepsilon = o(\varepsilon)$  by hypothesis we have

$$\begin{aligned}
D_1 &\leq c \varepsilon^{-\frac{3}{p^*}} \left\{ \varepsilon^2 + \varepsilon^{1/2}(1-\theta)\|\phi_1\|_\varepsilon + \varepsilon^{1/2}\theta\|\phi_2\|_\varepsilon \right\} \varepsilon^{1/2} \|\phi_1 - \phi_2\|_\varepsilon \\
&\leq c \varepsilon^{\frac{5}{2} - \frac{3}{p^*}} \|\phi_1 - \phi_2\|_\varepsilon
\end{aligned}$$

and the claim is proved since  $\frac{5}{2} - \frac{3}{p^*} > 0$  if  $p' > \frac{6}{5}$  that is true since  $p < 6$ . For  $n = 4$ , analogously we have, by (20)

$$\begin{aligned}
D_1 &\leq c \varepsilon^{-\frac{4}{p^*}} \{ \varepsilon^2 + \varepsilon(1-\theta)\|\phi_1\|_\varepsilon + \varepsilon\theta\|\phi_2\|_\varepsilon \} \varepsilon \|\phi_1 - \phi_2\|_\varepsilon \\
&\leq c \varepsilon^{3 - \frac{4}{p^*}} \|\phi_1 - \phi_2\|_\varepsilon
\end{aligned}$$

and  $3 - \frac{4}{p^*} > 0$  iff  $p' > \frac{4}{3}$  that is  $p < 4$ . □

We can now prove the main result of this section

*Proof of Proposition 11.* The proof is similar to Proposition 3.5 of [20], which we refer to for all details. We want to solve (29) by a fixed point argument. We define the operator

$$\begin{aligned} T_{\varepsilon,\xi} &: K_{\varepsilon,\xi}^\perp \rightarrow K_{\varepsilon,\xi}^\perp \\ T_{\varepsilon,\xi}(\phi) &= L_{\varepsilon,\xi}^{-1}(N_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi}S_{\varepsilon,\xi}(\phi)) \end{aligned}$$

By Remark 12  $T_{\varepsilon,\xi}$  is well defined and it holds

$$\begin{aligned} \|T_{\varepsilon,\xi}(\phi)\|_\varepsilon &\leq c(\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon + \|R_{\varepsilon,\xi}\|_\varepsilon + \|S_{\varepsilon,\xi}(\phi)\|_\varepsilon) \\ \|T_{\varepsilon,\xi}(\phi_1) - T_{\varepsilon,\xi}(\phi_2)\|_\varepsilon &\leq c(\|N_{\varepsilon,\xi}(\phi_1) - N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon + \|S_{\varepsilon,\xi}(\phi_1) - S_{\varepsilon,\xi}(\phi_2)\|_\varepsilon) \end{aligned}$$

for some suitable constant  $c > 0$ . By the mean value theorem (and by the properties of  $i^*$ ) we get

$$\|N_{\varepsilon,\xi}(\phi_1) - N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \leq c|f'(W_{\varepsilon,\xi} + \phi_2 + t(\phi_1 - \phi_2)) - f'(W_{\varepsilon,\xi})|_{\frac{p}{p-2},\varepsilon} \|\phi_1 - \phi_2\|_\varepsilon.$$

By [20], Remark 3.4 we have that  $|f'(W_{\varepsilon,\xi} + \phi_2 + t(\phi_1 - \phi_2)) - f'(W_{\varepsilon,\xi})|_{\frac{p}{p-2},\varepsilon} < 1$  provided  $\|\phi_1\|_\varepsilon$  and  $\|\phi_2\|_\varepsilon$  small enough. This, combined with (32) proves that there exists  $0 < L < 1$  such that  $\|T_{\varepsilon,\xi}(\phi_1) - T_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \leq L\|\phi_1 - \phi_2\|_\varepsilon$ .

We recall that by Lemma 12 we have

$$\begin{aligned} \|N_{\varepsilon,\xi}(\phi)\|_\varepsilon &\leq c(\|\phi\|_\varepsilon^2 + \|\phi\|_\varepsilon^{p-1}) \\ \|R_{\varepsilon,\xi}\|_\varepsilon &\leq \varepsilon^{1+\frac{n}{p^*}} = o(\varepsilon^2) \end{aligned}$$

This, combined with (31) gives us

$$\begin{aligned} \|T_{\varepsilon,\xi}(\phi)\|_\varepsilon &\leq c(\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon + \|R_{\varepsilon,\xi}\|_\varepsilon + \|S_{\varepsilon,\xi}(\phi)\|_\varepsilon) \\ &\leq c(\|\phi\|_\varepsilon^2 + \|\phi\|_\varepsilon^{p-1} + \varepsilon^{1+\frac{n}{p^*}} + c\varepsilon^2) \end{aligned}$$

So, there exists a positive constant  $C$  such that  $T_{\varepsilon,\xi}$  maps a ball of center 0 and radius  $C\varepsilon^2$  in  $K_{\varepsilon,\xi}^\perp$  into itself and it is a contraction. So there exists a fixed point  $\phi_{\varepsilon,\xi}$  with norm  $\|\phi_{\varepsilon,\xi}\|_\varepsilon \leq C\varepsilon^2$ .

The continuity of  $\phi_{\varepsilon,\xi}$  with respect to  $\xi$  is standard.  $\square$

#### 4. THE REDUCED FUNCTIONAL

In this section we define the reduced functional in a finite dimensional space and we solve equation (30). This leads us to the prove of main theorem.

We have introduced  $I_\varepsilon(u)$  in the introduction. We now define the reduced functional

$$\begin{aligned} \tilde{I}_\varepsilon &: \partial M \rightarrow \mathbb{R} \\ \tilde{I}_\varepsilon(\xi) &= I_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \end{aligned}$$

where  $\phi_{\varepsilon,\xi}$  is uniquely determined by Proposition 11.

**Lemma 14.** *Let  $\xi_0$  a critical point of  $\tilde{I}_\varepsilon$ , that is, if  $\xi = \xi(y) = \exp_{\xi_0}^\partial(y)$ ,  $y \in B^{n-1}(0, r)$ , then*

$$\left( \frac{\partial}{\partial y_h} \tilde{I}_\varepsilon(\xi(y)) \right)_{|y=0} = 0, \quad h = 1, \dots, n-1.$$

*Thus the function  $\phi_{\varepsilon,\xi} + W_{\varepsilon,\xi}$  solves equation (30).*

*Proof.* The proof of this lemma is just a computation.  $\square$

**Lemma 15.** *It holds*

$$\tilde{I}_\varepsilon(\xi) = C - \varepsilon H(\xi) + o(\varepsilon)$$

$C^1$  uniformly with respect to  $\xi \in \partial M$  as  $\varepsilon$  goes to zero. Here  $H(\xi)$  is the mean curvature of the boundary  $\partial M$  at  $\xi$ .

To prove Lemma we study the asymptotic expansion of  $\tilde{I}_\varepsilon(\xi)$  with respect to  $\varepsilon$ . We recall the result contained in [13].

*Remark 16.* It holds

$$(33) \quad \begin{aligned} \tilde{J}_\varepsilon(\xi) &:= J_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) = J_\varepsilon(W_{\varepsilon,\xi}) + o(\varepsilon) \\ &= C - \varepsilon \alpha H(\xi) + o(\varepsilon) \end{aligned}$$

$C^1$  uniformly with respect to  $\xi \in \partial M$  as  $\varepsilon$  goes to zero, where

$$\begin{aligned} C &:= \int_{\mathbb{R}_+^n} \frac{1}{2} |\nabla U(z)|^2 + \frac{1}{2} U^2(z) - \frac{1}{p} U^p(z) dz \\ \alpha &:= \frac{(n-1)}{2} \int_{\mathbb{R}_+^n} \left( \frac{U'(|z|)}{|z|} \right)^2 z_n^3 dz \end{aligned}$$

In light of this result, it remains to estimate the coupling functional  $G_\varepsilon$  to prove Lemma 15. We split this proof in several lemmas.

**Lemma 17.** *It holds*

$$(34) \quad G_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - G_\varepsilon(W_{\varepsilon,\xi}) = o(\varepsilon)$$

$$(35) \quad [G'_\varepsilon(W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) - G'_\varepsilon(W_{\varepsilon,\xi_0})] \left[ \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right] = o(\varepsilon)$$

$$(36) \quad G'_\varepsilon(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \left[ \frac{\partial}{\partial y_h} \phi_{\varepsilon,\xi(y)} \right] = o(\varepsilon)$$

*Proof.* Let us prove (34). We have (for some  $\theta \in [0, 1]$ )

$$\begin{aligned} &G_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - G_\varepsilon(W_{\varepsilon,\xi}) \\ &= \frac{1}{\varepsilon^n} \int_M \left[ \psi(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})^2 - \psi(W_{\varepsilon,\xi}) (W_{\varepsilon,\xi})^2 \right] \\ &= \frac{1}{\varepsilon^n} \int_M \psi'(W_{\varepsilon,\xi} + \theta \phi_{\varepsilon,\xi}) [\phi_{\varepsilon,\xi}] (W_{\varepsilon,\xi})^2 \\ &\quad + \frac{1}{\varepsilon^n} \int_M \psi(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) (2\phi_{\varepsilon,\xi} W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}^2) := I_1 + I_2. \end{aligned}$$

By Lemma 10 and Remark 8 we have

$$\begin{aligned} I_1 &\leq \frac{1}{\varepsilon^n} \left( \int_M (\psi'(W_{\varepsilon,\xi} + \theta \phi_{\varepsilon,\xi}) [\phi_{\varepsilon,\xi}])^2 d\mu_g \right)^{\frac{1}{2}} \left( \int_M W_{\varepsilon,\xi}^4 d\mu_g \right)^{\frac{1}{2}} \\ &\leq \frac{\varepsilon^{\frac{n}{2}}}{\varepsilon^n} \|\psi'(W_{\varepsilon,\xi} + \theta \phi_{\varepsilon,\xi}) [\phi_{\varepsilon,\xi}]\|_H \|W_{\varepsilon,\xi}\|_{\varepsilon,2}^2 \\ &\leq \varepsilon^{-\frac{n}{2}} (\varepsilon^2 \|\phi_{\varepsilon,\xi}\|_H + \|\phi_{\varepsilon,\xi}\|_H^2) \leq \varepsilon^{\frac{9-n}{2}} = o(\varepsilon). \end{aligned}$$

since  $\|\phi_{\varepsilon,\xi}\|_H \leq \varepsilon^{1/2} \|\phi_{\varepsilon,\xi}\|_\varepsilon \leq \varepsilon^{5/2}$  by Proposition 11.

For  $I_2$  we have, by (24) and Remark 8 in a similar way we get

$$\begin{aligned}
I_2 &\leq \frac{1}{\varepsilon^n} \left( \int_M \psi^2 (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) d\mu_g \right)^{\frac{1}{2}} \left( \int_M \phi_{\varepsilon,\xi}^4 d\mu_g \right)^{\frac{1}{2}} \\
&\quad + \frac{1}{\varepsilon^n} \left( \int_M \psi^3 (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) d\mu_g \right)^{\frac{1}{3}} \left( \int_M \phi_{\varepsilon,\xi}^3 d\mu_g \right)^{\frac{1}{3}} \left( \int_M W_{\varepsilon,\xi}^3 d\mu_g \right)^{\frac{1}{3}} \\
&\leq \frac{1}{\varepsilon^n} \|\psi (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\|_H \|\phi_{\varepsilon,\xi}\|_H^2 + \\
&\quad + \frac{\varepsilon^{\frac{n}{3}}}{\varepsilon^n} \|\psi (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\|_H \|\phi_{\varepsilon,\xi}\|_H |W_{\varepsilon,\xi}|_{\varepsilon,3} \\
&\leq \varepsilon^{-n+\frac{n+2}{2}+5} + \varepsilon^{-\frac{2}{3}n+\frac{n+2}{2}+\frac{5}{2}} = \varepsilon^{\frac{12-n}{2}} + \varepsilon^{\frac{21-n}{6}} = o(\varepsilon)
\end{aligned}$$

since  $n = 3, 4$ . Then (34) follows.

Let us prove (35). Since  $0 \leq \psi \leq 1/q$  we have

$$\begin{aligned}
&[G'_\varepsilon (W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) - G'_\varepsilon (W_{\varepsilon,\xi_0})] \left[ \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right] \\
&\leq \left| \frac{c}{\varepsilon^n} \int_M \{ \psi (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - \psi (W_{\varepsilon,\xi}) \} W_{\varepsilon,\xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right| \\
&\quad + \left| \frac{c}{\varepsilon^n} \int_M \{ \psi^2 (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - \psi^2 (W_{\varepsilon,\xi}) \} W_{\varepsilon,\xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right| \\
&\quad + \left| \frac{c}{\varepsilon^n} \int_M \psi (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \phi_{\varepsilon,\xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right| \\
&\quad + \left| \frac{c}{\varepsilon^n} \int_M \psi^2 (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \phi_{\varepsilon,\xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right| \\
&\leq \left| \frac{c}{\varepsilon^n} \int_M \{ \psi (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - \psi (W_{\varepsilon,\xi}) \} W_{\varepsilon,\xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right| \\
&\quad + \left| \frac{c}{\varepsilon^n} \int_M \psi (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \phi_{\varepsilon,\xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right| \\
&\leq \left| \frac{c}{\varepsilon^n} \int_M \{ \psi' (W_{\varepsilon,\xi} + \theta \phi_{\varepsilon,\xi}) [\phi_{\varepsilon,\xi}] \} W_{\varepsilon,\xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right| \\
&\quad + \left| \frac{c}{\varepsilon^n} \int_M \psi (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \phi_{\varepsilon,\xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right| := D_1 + D_2
\end{aligned}$$

for some  $0 < \theta < 1$ .

By Lemma 10, Remark 8, recalling that  $\|\phi_{\varepsilon,\xi}\|_H \leq \varepsilon^{1/2}\|\phi_{\varepsilon,\xi}\|_H \leq \varepsilon^{5/2}$  and that  $\left\|\frac{\partial}{\partial y_n}W_{\varepsilon,\xi(y)}\right\|_\varepsilon = O\left(\frac{1}{\varepsilon}\right)$  (cfr. eq (38)) we have

$$\begin{aligned} D_1 &\leq \frac{c}{\varepsilon^n} \left( \int_M \{\psi'(W_{\varepsilon,\xi} + \theta\phi_{\varepsilon,\xi})[\phi_{\varepsilon,\xi}]\}^3 \right)^{\frac{1}{3}} \left( \int_M W_{\varepsilon,\xi(y)}^3 \right)^{\frac{1}{3}} \left( \int_M \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)^3 \right)^{\frac{1}{3}} \\ &\leq c \frac{\varepsilon^{\frac{2}{3}n}}{\varepsilon^n} \|\psi'(W_{\varepsilon,\xi} + \theta\phi_{\varepsilon,\xi})[\phi_{\varepsilon,\xi}]\|_H \|W_{\varepsilon,\xi(y)}\|_\varepsilon \left\| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right\|_\varepsilon \\ &\leq c\varepsilon^{-1-\frac{n}{3}} \|\psi'(W_{\varepsilon,\xi} + \theta\phi_{\varepsilon,\xi})[\phi_{\varepsilon,\xi}]\|_H \leq c\varepsilon^{-1-\frac{n}{3}} \|\phi_{\varepsilon,\xi}\|_H \{\varepsilon^2 + \|\phi_{\varepsilon,\xi}\|_H\} \\ &\leq c\varepsilon^{-1-\frac{n}{3}} \varepsilon^{\frac{5}{2}} \varepsilon^2 = c\varepsilon^{\frac{7}{2}-\frac{n}{3}} = o(\varepsilon). \end{aligned}$$

In a similar way, using (24) and the above estimates we get

$$\begin{aligned} D_2 &\leq \frac{c}{\varepsilon^n} \left( \int_M \psi^3(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \right)^{\frac{1}{3}} \left( \int_M \phi_{\varepsilon,\xi_0}^3 \right)^{\frac{1}{3}} \left( \int_M \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)^3 \right)^{\frac{1}{3}} \\ &\leq c \frac{\varepsilon^{\frac{n}{3}}}{\varepsilon^n} \|\psi(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\|_H \|\phi_{\varepsilon,\xi}\|_H \left\| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right\|_\varepsilon \\ &\leq c\varepsilon^{-\frac{2}{3}n-1} \varepsilon^{\frac{5}{2}} \|\psi(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\|_H \leq c\varepsilon^{-\frac{2}{3}n+\frac{3}{2}} \varepsilon^{\frac{n+2}{n}} (1 + \|\phi_{\varepsilon,\xi}\|_\varepsilon) \\ &\leq c\varepsilon^{\frac{15-n}{6}} = o(\varepsilon) \end{aligned}$$

and (35) is proved.

The prove of (36) requires to estimate that

$$(37) \quad I := \left| \frac{1}{\varepsilon^n} \int_M [q^2 \psi^2(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - 2q\psi(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})] (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) Z_{\varepsilon,\xi(y)}^l \right| = o(\varepsilon),$$

where the functions  $Z_{\varepsilon,\xi(y)}^l$  are defined in (27). By (37) it is possible to complete the proof the lemma, with the same arguments the proof of (5.10) in [20], which we refer to for the missing details. To prove (37), since  $0 < \psi < 1/q$ , we get, as before,

$$\begin{aligned} I &\leq \left| \frac{c}{\varepsilon^n} \int_M \psi(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) Z_{\varepsilon,\xi(y)}^l \right| \leq \\ &\leq c \frac{\varepsilon^{\frac{n+2}{2}}}{\varepsilon^n} \left( \int_M \psi^{2^*}(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \right)^{\frac{1}{2^*}} \left( \frac{1}{\varepsilon^n} \int_M (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})^{\frac{4n}{n+2}} \right)^{\frac{n+2}{4n}} \\ &\quad \times \left( \frac{1}{\varepsilon^n} \int_M (Z_{\varepsilon,\xi(y)}^l)^{\frac{4n}{n+2}} \right)^{\frac{n+2}{4n}} \\ &\leq c\varepsilon^{\frac{2-n}{2}} \|\psi(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})\|_H |W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}|_{\varepsilon, \frac{4n}{n+2}} |Z_{\varepsilon,\xi(y)}^l|_{\varepsilon, \frac{4n}{n+2}}. \end{aligned}$$

Arguing as in Remark 8, we have that  $|Z_{\varepsilon,\xi(y)}^l|_{\varepsilon, \frac{4n}{n+2}} \rightarrow |\varphi^l|_{\frac{4n}{n+2}}$ , so, by (24) we obtain

$$I \leq c\varepsilon^{\frac{2-n}{2}} \varepsilon^{\frac{n+2}{2}} = c\varepsilon^2.$$

This concludes the proof.  $\square$

**Lemma 18.** *It holds that*

$$G_\varepsilon(W_{\varepsilon,\xi}) := \frac{1}{\varepsilon^n} \int_M \psi(W_{\varepsilon,\xi}) W_{\varepsilon,\xi}^2 d\mu_g = o(\varepsilon)$$

$C^1$ -uniformly with respect to  $\xi \in M$  as  $\varepsilon$  goes to zero.

*Proof.* At first we have, by Remark 8 and by (23)

$$G_\varepsilon(W_{\varepsilon,\xi}) \leq c \frac{1}{\varepsilon^n} \left( \int_M \psi^3(W_{\varepsilon,\xi}) \right)^{\frac{1}{3}} \left( \int_M W_{\varepsilon,\xi}^3 \right)^{\frac{2}{3}} \leq c \frac{1}{\varepsilon^n} \varepsilon^{\frac{n+2}{2}} \varepsilon^{\frac{2}{3}n} = c \varepsilon^{\frac{n}{6}+1} = o(\varepsilon).$$

We want now to prove the  $C^1$  convergence, id est, if  $\xi(y) = \exp_\xi(y)$  for  $y \in B(0, r)$ , we will prove that

$$\frac{\partial}{\partial y_h} G_\varepsilon(W_{\varepsilon,\xi}) \Big|_{y=0} = \frac{2}{\varepsilon^n} \int_M (2q\psi(W_{\varepsilon,\xi}) - q^2\psi^2(W_{\varepsilon,\xi})) W_{\varepsilon,\xi} \left[ \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(h)} \Big|_{y=0} \right] d\mu_g$$

for  $h = 1, \dots, n-1$ . Since  $0 < \psi < 1/q$ , immediately we have

$$\left| \frac{\partial}{\partial y_h} G_\varepsilon(W_{\varepsilon,\xi}) \Big|_{y=0} \right| \leq c \left| \frac{1}{\varepsilon^n} \int_M \psi(W_{\varepsilon,\xi(y)}) W_{\varepsilon,\xi(y)} \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(h)} \Big|_{y=0} d\mu_g \right|$$

Set  $I_1$  the quantity inside the absolute value at the r.h.s. of the above equation. Using the Fermi coordinates and the previous estimates we get

$$\begin{aligned} \frac{1}{\varepsilon^2} I_1(\varepsilon, \xi) &= \int_{\mathbb{R}_+^n} \frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2} 2U(z) \chi_R(\varepsilon z) |g_\xi(\varepsilon z)|^{1/2} \times \\ &\quad \times \left\{ \sum_{k=1}^3 \left[ \frac{1}{\varepsilon} \frac{\partial U(z)}{\partial z_k} \chi_R(\varepsilon z) + U(z) \frac{\partial \chi_R(\varepsilon z)}{\partial z_k} \right] \frac{\partial}{\partial y_h} \mathcal{H}_k(0, \exp_\xi(\varepsilon z)) \right\} dz. \end{aligned}$$

where  $\mathcal{H}_k(x, y)$  is introduced in Definition (22). Since  $|g_\xi(\varepsilon z)|^{1/2} = 1 + O(\varepsilon|z|)$  and by Lemma 24 we have

$$\begin{aligned} I_1(\varepsilon, \xi) &= 2\varepsilon \int_{\mathbb{R}_+^n} \tilde{v}_{\varepsilon,\xi}(z) U(z) \frac{\partial U(z)}{\partial z_h} \chi_R^2(\varepsilon z) dz + o(\varepsilon) \\ &= 2\varepsilon \int_{\mathbb{R}_+^n} \tilde{v}_{\varepsilon,\xi}(z) U(z) \frac{\partial U(z)}{\partial z_h} dz + o(\varepsilon) \end{aligned}$$

By Lemma 20 we have that  $\left\{ \frac{1}{\varepsilon_n^2} \tilde{v}_{\varepsilon_n,\xi} \right\}_n$  converges to  $\gamma$  weakly in  $L^{2^*}(\mathbb{R}_+^n)$ , so we have

$$I_1(\varepsilon, \xi) = 2\varepsilon \int_{\mathbb{R}^n} \gamma U(z) \frac{\partial U(z)}{\partial z_h} dz + o(\varepsilon)$$

where  $h = 1, \dots, n-1$ . Finally, we have that  $\int_{\mathbb{R}^n} \gamma(z) U(z) \frac{\partial U(z)}{\partial z_h} dz = 0$  because both  $\gamma$  (see Remark 21) and  $U$  are symmetric with respect to  $z_1, \dots, z_{n-1}$  while  $\frac{\partial U(z)}{\partial z_h}$  is antisymmetric. This concludes the proof.  $\square$

We can now prove Lemma 15.

*Proof of Lemma 15.* We want to estimate

$$I_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) = J_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) + \frac{\omega^2}{2} G_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}),$$

By Remark 16 we have that

$$J_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) = J_\varepsilon(W_{\varepsilon,\xi}) + o(\varepsilon) = C - \varepsilon \alpha H(\xi) + o(\varepsilon)$$

$C^1$  uniformly with respect to  $\xi \in \partial M$  as  $\varepsilon$  goes to zero. Moreover by Lemma 17 and by Lemma 18 we have that

$$G_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) = o(\varepsilon)$$

$C^1$  uniformly with respect to  $\xi \in \partial M$  and this concludes the proof.  $\square$

**4.1. Sketch of the proof of Theorem 2.** In section 3, Proposition 11 we found a function  $\phi_{\varepsilon,\xi}$  solving (29). By Lemma 14 we can solve (30) once we have a critical point of functional  $\tilde{I}_\varepsilon$ . At this point by Lemma 15 and by definition of  $C^1$  stable critical point (Def. 1) we can complete the proof.

#### APPENDIX A. TECHNICAL LEMMAS

**Lemma 19.** *There exists  $\varepsilon_0 > 0$  and  $c > 0$  such that, for any  $\xi_0 \in \partial M$  and for any  $\varepsilon \in (0, \varepsilon_0)$  it holds*

$$(38) \quad \left\| \frac{\partial}{\partial y_h} Z_{\varepsilon,\xi(y)}^l \right\|_\varepsilon = O\left(\frac{1}{\varepsilon}\right), \quad \left\| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right\|_\varepsilon = O\left(\frac{1}{\varepsilon}\right),$$

for  $h = 1, \dots, n-1$ ,  $l = 1, \dots, n$

**Lemma 20.** *Let us consider the functions*

$$\tilde{v}_{\varepsilon,\xi}(z) = \begin{cases} \psi(W_{\varepsilon,\xi}) \left( \Psi_\xi^\partial(\varepsilon z) \right) & \text{for } z \in D^+(R/\varepsilon) \\ 0 & \text{for } z \in \mathbb{R}^3 \setminus D^+(R/\varepsilon) \end{cases}$$

Where  $D^+(r/\varepsilon) = \{z = (\bar{z}, z_n), \bar{z} \in \mathbb{R}^{n-1}, |\bar{z}| < r/\varepsilon, 0 \leq z_n < R/\varepsilon\}$ . Then there exists a constant  $c > 0$  such that

$$\|\tilde{v}_{\varepsilon,\xi}(z)\|_{L^{2^*}(\mathbb{R}_+^n)} \leq c\varepsilon^2.$$

Furthermore, take a sequence  $\varepsilon_n \rightarrow 0$ , up to subsequences,  $\left\{ \frac{1}{\varepsilon_n^2} \tilde{v}_{\varepsilon_n,\xi} \right\}_n$  converges weakly in  $L^{2^*}(\mathbb{R}_+^n)$  as  $\varepsilon$  goes to 0 to a function  $\gamma \in D^{1,2}(\mathbb{R}^3)$ . The function  $\gamma$  solves, in a weak sense, the equation

$$(39) \quad -\Delta \gamma = qU^2 \text{ in } \mathbb{R}_+^n$$

*Proof.* We prove the Lemma for Problem (1), being the Problem (2) completely analogous. By definition of  $\tilde{v}_{\varepsilon,\xi}(z)$  and by (1) we have, for all  $z \in D^+(r/\varepsilon)$ ,

$$(40) \quad - \sum_{ij} \partial_j \left( |g_\xi(\varepsilon z)|^{1/2} g_\xi^{ij}(\varepsilon z) \partial_i \tilde{v}_{\varepsilon,\xi}(z) \right) = \\ = \varepsilon^2 |g_\xi(\varepsilon z)|^{1/2} \{ qU^2(z) \chi_r^2(\varepsilon z) - [1 + q^2 U^2(z) \chi_R^2(\varepsilon z)] \tilde{v}_{\varepsilon,\xi}(z) \}$$

By (40), and remarking that  $\tilde{v}_{\varepsilon,\xi}(z) \geq 0$  we have

$$\begin{aligned} \|\tilde{v}_{\varepsilon,\xi}(z)\|_{D^{1,2}(D^+(r/\varepsilon))}^2 &\leq C \int_{D^+(R/\varepsilon)} |g_\xi(\varepsilon z)|^{1/2} g_\xi^{ij}(\varepsilon z) \partial_i \tilde{v}_{\varepsilon,\xi}(z) \partial_j \tilde{v}_{\varepsilon,\xi}(z) dz \\ &= C \varepsilon^2 \int_{D^+(R/\varepsilon)} |g_\xi(\varepsilon z)|^{1/2} \{ qU^2(z) \chi_R^2(\varepsilon z) \tilde{v}_{\varepsilon,\xi}(z) - [1 + q^2 U^2(z) \chi_R^2(\varepsilon z)] \tilde{v}_{\varepsilon,\xi}^2(z) \} dz \\ &\leq C \varepsilon^2 \int_{D^+(R/\varepsilon)} |g_\xi(\varepsilon z)|^{1/2} qU^2(z) \chi_R^2(\varepsilon |z|) \tilde{v}_{\varepsilon,\xi}(z) dz \\ &\leq C \varepsilon^2 \|\tilde{v}_{\varepsilon,\xi}(z)\|_{L^{2^*}(D^+(R/\varepsilon))} \|U\|_{L^{\frac{4n}{n+2}}}^2 \leq C \varepsilon^2 \|\tilde{v}_{\varepsilon,\xi}(z)\|_{D^{1,2}(D^+(R/\varepsilon))} \end{aligned}$$

Thus we have

$$(41) \quad \|\tilde{v}_{\varepsilon,\xi}(z)\|_{D^{1,2}(D^+(R/\varepsilon))} \leq C \varepsilon^2 \text{ and } |\tilde{v}_{\varepsilon,\xi}(z)|_{L^{2^*}(\mathbb{R}_+^n)} \leq C \varepsilon^2.$$

By (41), if  $\varepsilon_n$  is a sequence which goes to zero, the sequence  $\left\{ \frac{1}{\varepsilon_n^2} \tilde{v}_{\varepsilon_n,\xi} \right\}_n$  is bounded in  $L^{2^*}(\mathbb{R}_+^n)$ . Then, up to subsequence,  $\left\{ \frac{1}{\varepsilon_n^2} \tilde{v}_{\varepsilon_n,\xi} \right\}_n$  converges to some  $\tilde{\gamma} \in L^{2^*}(\mathbb{R}_+^n)$  weakly in  $L^{2^*}(\mathbb{R}_+^n)$ .

Moreover, by (40), for any  $\varphi \in C_0^\infty(\mathbb{R}_+^n)$ , it holds

$$(42) \quad \int_{\text{supp } \varphi} \sum_{ij} |g_\xi(\varepsilon z)|^{1/2} g_\xi^{ij}(\varepsilon z) \partial_i \frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon_n^2} \partial_j \varphi(z) dz = \\ \int_{\text{supp } \varphi} \{ qU^2(z) \chi_r^2(\varepsilon |z|) - [1 + q^2 U^2(z) \chi_R^2(\varepsilon z)] \tilde{v}_{\varepsilon,\xi}(z) \} |g_\xi(\varepsilon z)|^{1/2} \varphi(z) dz.$$

Consider now the functions

$$v_{\varepsilon,\xi}(z) := \psi(W_{\varepsilon,\xi}) (\Psi_\xi^\partial(\varepsilon z)) \chi_R(\varepsilon z) = \tilde{v}_{\varepsilon,\xi}(z) \chi_r(\varepsilon z) \text{ for } z \in \mathbb{R}_+^n.$$

We have immediately that  $v_{\varepsilon,\xi}(z)$  is bounded in  $D^{1,2}(\mathbb{R}_+^n)$ , thus the sequence  $\left\{ \frac{1}{\varepsilon_n^2} v_{\varepsilon_n,\xi} \right\}_n$  converges to some  $\gamma \in D^{1,2}(\mathbb{R}^3)$  weakly in  $D^{1,2}(\mathbb{R}_+^n)$  and in  $L^{2^*}(\mathbb{R}_+^n)$ . Finally, for any compact set  $K \subset \mathbb{R}_+^n$  eventually  $v_{\varepsilon_n,\xi} \equiv \tilde{v}_{\varepsilon_n,\xi}$  on  $K$ . So it is easy to see that  $\tilde{\gamma} = \gamma$ .



We recall that  $|g_\xi(\varepsilon z)|^{1/2} = 1 + O(\varepsilon|z|)$  and  $g_\xi^{ij}(\varepsilon z) = \delta_{ij} + O(\varepsilon|z|)$  so, by the weak convergence of  $\left\{\frac{1}{\varepsilon_n^2}v_{\varepsilon_n,\xi}\right\}_n$  in  $D^{1,2}(\mathbb{R}_+^n)$ , for any  $\varphi \in C_0^\infty(\mathbb{R}_+^n)$  we get

$$\begin{aligned}
 (43) \quad & \int_{\text{supp } \varphi} \sum_{ij} |g_\xi(\varepsilon_n z)|^{1/2} g_\xi^{ij}(\varepsilon_n z) \partial_i \frac{\tilde{v}_{\varepsilon_n,\xi}(z)}{\varepsilon_n^2} \partial_j \varphi(z) dz \\
 &= \int_{\text{supp } \varphi} \sum_{ij} |g_\xi(\varepsilon_n z)|^{1/2} g_\xi^{ij}(\varepsilon_n z) \partial_i \frac{v_{\varepsilon_n,\xi}(z)}{\varepsilon_n^2} \partial_j \varphi(z) dz \\
 &\rightarrow \int_{\mathbb{R}^3} \sum_i \partial_i \gamma(z) \partial_i \varphi(z) dz \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Thus by (42) and by (43) and because  $\left\{\frac{1}{\varepsilon_n^2}\tilde{v}_{\varepsilon_n,\xi}\right\}_n$  converges to  $\gamma$  weakly in  $L^{2^*}(\mathbb{R}_+^n)$  we get

$$\int_{\mathbb{R}_+^n} \sum_i \partial_i \gamma(z) \partial_i \varphi(z) dz = q \int_{\mathbb{R}_+^n} U^2(z) \varphi(z) dz \text{ for all } \varphi \in C_0^\infty(\mathbb{R}_+^n).$$

So, finally, up to subsequences,  $\left\{\frac{1}{\varepsilon_n^2}\tilde{v}_{\varepsilon_n,\xi}\right\}_n$  converges to  $\gamma$ , weakly in  $L^{2^*}(\mathbb{R}_+^n)$  and the function  $\gamma \in D^{1,2}(\mathbb{R}_+^n)$  is a weak solution of  $-\Delta\gamma = qU^2$  in  $\mathbb{R}_+^n$ .  $\square$

*Remark 21.* We remark that  $\gamma$  is positive and decays exponentially at infinity with its first derivative because it solves  $-\Delta\gamma = qU^2$  in  $\mathbb{R}_+^n$ . Moreover its is symmetric with respect to the first  $n-1$  variables.

**Definition 22.** Let  $\xi_0 \in \partial M$ . We introduce the functions  $\mathcal{E}$  and  $\tilde{\mathcal{E}}$  as follows.

$$\mathcal{E}(y, x) = \left(\exp_{\xi(y)}^\partial\right)^{-1}(x) = \left(\exp_{\exp_{\xi_0}^\partial y}^\partial\right)^{-1}(\exp_{\xi_0}^\partial \bar{\eta}) = \tilde{\mathcal{E}}(y, \bar{\eta})$$

where  $x, \xi(y) \in \partial M$ ,  $y, \bar{\eta} \in B(0, R) \subset \mathbb{R}^{n-1}$  and  $\xi(y) = \exp_{\xi_0}^\partial y$ ,  $x = \exp_{\xi_0}^\partial \bar{\eta}$ . Using Fermi coordinates, in a similar way we define

$$\mathcal{H}(y, x) = \left(\psi_{\xi(y)}^\partial\right)^{-1}(x) = \left(\psi_{\exp_{\xi_0}^\partial y}^\partial\right)^{-1}(\psi_{\xi_0}^\partial(\bar{\eta}, \eta_n)) = \tilde{\mathcal{H}}(y, \bar{\eta}, \eta_n) = (\tilde{\mathcal{E}}(y, \bar{\eta}), \eta_n)$$

where  $x \in M$ ,  $\eta = (\bar{\eta}, \eta_n)$ , with  $\bar{\eta} \in B(0, R) \subset \mathbb{R}^{n-1}$  and  $0 \leq \eta_n < R$ ,  $\xi(y) = \exp_{\xi_0}^\partial y \in \partial M$  and  $x = \psi_{\xi_0}^\partial(\eta)$ .

**Lemma 23.** *It holds*

$$\frac{\partial \tilde{\mathcal{E}}_k}{\partial y_j}(0, 0) = -\delta_{jk} \text{ for } j, k = 1, \dots, n-1$$

*Proof.* We recall that  $\tilde{\mathcal{E}}(y, \bar{\eta}) = \left(\exp_{\xi(y)}^\partial\right)^{-1}(\exp_{\xi_0}^\partial \bar{\eta})$ . Let us introduce, for  $y, \bar{\eta} \in B(0, R) \subset \mathbb{R}^{n-1}$

$$\begin{aligned}
 F(y, \bar{\eta}) &= \left(\exp_{\xi_0}^\partial\right)^{-1}\left(\exp_{\xi(y)}^\partial(\bar{\eta})\right) \\
 \Gamma(y, \bar{\eta}) &= (y, F(y, \bar{\eta})).
 \end{aligned}$$

We notice that  $\Gamma^{-1} = (y, \tilde{\mathcal{E}}(y, \bar{\eta}))$ . We can easily compute the derivative of  $\Gamma$ . We have

$$\Gamma'(\hat{y}, \hat{\eta})[\tilde{y}, \tilde{\eta}] = \begin{pmatrix} \text{Id}_{\mathbb{R}^{n-1}} & 0 \\ F'_y(\hat{y}, \hat{\eta}) & F'_\eta(\hat{y}, \hat{\eta}) \end{pmatrix} \begin{pmatrix} \tilde{y} \\ \tilde{\eta} \end{pmatrix},$$

thus

$$(\Gamma^{-1})'(\hat{y}, \hat{\eta})[\tilde{y}, \tilde{\eta}] = \begin{pmatrix} \text{Id}_{\mathbb{R}^{n-1}} & 0 \\ -(F'_\eta(\hat{y}, \hat{\eta}))^{-1} F'_y(\hat{y}, \hat{\eta}) & (F'_\eta(\hat{y}, \hat{\eta}))^{-1} \end{pmatrix} \begin{pmatrix} \tilde{y} \\ \tilde{\eta} \end{pmatrix}$$

Now, by direct computation we have that

$$F'_\eta(0, \bar{\eta}) = \text{Id}_{\mathbb{R}^{n-1}} \text{ and } F'_y(\hat{y}, 0) = \text{Id}_{\mathbb{R}^{n-1}},$$

$$\text{so } \frac{\partial \tilde{\mathcal{E}}_k}{\partial y_j}(0, 0) = \left( - (F'_\eta(0, 0))^{-1} F'_y(0, 0) \right)_{jk} = -\delta_{jk}. \quad \square$$

**Lemma 24.** *We have that*

$$\begin{aligned} \tilde{\mathcal{H}}(0, \bar{\eta}, \eta_n) &= (\bar{\eta}, \eta_n) \text{ for } \bar{\eta} \in \mathbb{R}^{n-1}, \eta_n \in \mathbb{R}_+ \\ \frac{\partial \tilde{\mathcal{H}}_k}{\partial y_j}(0, 0, \eta_n) &= -\delta_{jk} \text{ for } j, k = 1, \dots, n-1, \eta_n \in \mathbb{R}_+ \\ \frac{\partial \tilde{\mathcal{H}}_n}{\partial y_j}(y, \bar{\eta}, \eta_n) &= 0 \text{ for } j = 1, \dots, n-1, y, \bar{\eta} \in \mathbb{R}^{n-1}, \eta_n \in \mathbb{R}_+ \end{aligned}$$

*Proof.* The first two claim follows immediately by Definition 22 and Lemma 23. For the last claim, observe that  $\tilde{\mathcal{H}}_k(y, \bar{\eta}, \eta_n) = \tilde{\mathcal{E}}_k(y, \bar{\eta})$  which does not depends on  $\eta_n$  as well as its derivatives.  $\square$

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